CONVERGENCE TO EXTREMAL PROCESSES IN RANDOM ENVIRONMENTS AND EXTREMAL AGEING IN SK MODELS

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ABSTRACT. This paper extends recent results on ageing in mean field spin glasses on short time scales, obtained by Ben Arous and Gün [2] in law with respect to the environment, to results that hold almost surely, respectively in probability, with respect to the environment. It is based on the methods put forward in [9, 8] and naturally complements [6].

1. INTRODUCTION AND MAIN RESULTS

Spin glasses have, for the last decades, presented some of the most interesting challenges to probability theory. Even mean-field models have prompted a 1000 page monograph [16, 17] by one of the most eminent probabilists of our time. Despite these efforts and remarkable and unexpected progress, a full understanding of the equilibrium problem, i.e. a full description of the asymptotic geometry of the Gibbs measures, is still outstanding. In this situation it is somewhat surprising that certain properties of their dynamics have been prone to rigorous analysis, at least for some limited choices of the dynamics. The reason for this is that interesting aspects of the dynamics occur on time-scales that are far shorter than those of equilibration, and experiments made with spin glasses usually test the behaviour of the probe on such time scales. Indeed, equilibration is expected to take so long as to become inaccessible to real experiments. The physically interesting issue is thus that of *ageing* [4, 5], a property of time-time correlation functions that characterizes the slow decay to equilibrium characteristic for these systems.

The mathematical analysis has revealed an universal mechanism behind this phenomenon: the convergence of the *clock-process*, that relates the physical time to the number of "moves" of the process, to an α-stable subordinator (increasing Lévy process) under proper rescaling. The parameter α can be thought of as an *effective temperature*, that depends both on the *physical temperature* and the *time scale* considered. This has been proven for p-spin Sherrington-Kirkpatrick (SK) models for time scales of the order $\exp(\beta\gamma n)$ (where n is the number of sites in the system) with $0 < \gamma < \min\big(\beta, \zeta(p)\big)$, where $\zeta(p)$ is an increasing function of p such that $\zeta(3) > 0$ and $\lim_{p\uparrow\infty} \zeta(p) = 2\ln 2$. Such a result was obtained first in [1] *in law* with respect to the random environment, and was later extended in [6] to almost sure (resp. in probability, for $p = 3, 4$) results. The progress in the latter paper was possible to a fresh view on the convergence of clock processes, introduced and illustrated in two papers [9, 8]. They view the clock process as a

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sum of *dependent* random variables with a *random distribution*, and then employ convenient convergence criteria, obtained by Durrett and Resnick [7] a long time ago, to prove convergence. This is explained in more detail below.

The conditions on the admissible time scales in these results have two reasons. First, it emerges that $\alpha = \gamma/\beta$, so one of the conditions is simply that $\alpha \in (0,1)$. The upper bound $\gamma < \zeta(p)$ ensures that there will be no strong long-distance correlations, meaning that the systems has not had time to discover the full correlation structure of the random environment. This condition is thus the stricter the smaller p is, since correlations become weaker as p increases.

A natural questions to ask is what happens on time-scales that are sub-exponential in the volume n? This question was first addressed in a recent paper by Ben Arous and Gün [2]. This situation would correspond formally to $\alpha = 0$, but 0-stable subordinators do not exist, so some new phenomenon has to appear. Indeed, Ben Arous and Gün showed that the limiting objects appearing here are the so-called *extremal processes*. In the theory of sums of heavy tailed random variables this idea goes back to Kasahara [10] who showed that by applying non-linear transformations to the sums of α_n-stable r.v.'s with $\alpha_n \downarrow 0$, extremal processes arise as limit processes. This program was implemented for clock processes by Ben Arous and Gün using the approach of [1] to handle the problems of dependence of the random variables involved. As a consequence, their results are again in law with respect to the random environment. An interesting aspect of this work was that, due to the very short time scales considered, the case $p = 2$, i.e. the original SK model, is also covered, whereas this is not the case for exponential times scales.

In the present paper we show that by proceeding along the line of [6], one can extend the results of Ben Arous and Gün to *quenched* results, holding for given random environments almost surely (if $p > 4$) resp. in probability (if $2 \leq p \leq 4$). In fact, the result we present for the SK models is an application of an abstract result we establish, and that can be applied presumably to all models where ageing was analysed, on the approriate time scales.

Before stating our results, we begin by a concise description of the class of models we consider.

1.1. **Markov jump processes in random environments.** Let us describe the general setting of *Markov jump processes* in random environments that we consider here. Let $G_n(\mathcal{V}_n, \mathcal{L}_n)$ be a sequence of loop-free graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{L}_n$. The *random environment* is a family of positive random variables, $\tau_n(x), x \in \mathcal{V}_n$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that in the most interesting situations the τ_n's are correlated random variables.

On $\mathcal{V}_n$ we consider a discrete time Markov chain J_n with initial distribution μ_n, transition probabilities $p_n(x,y)$, and transition graph $G_n(\mathcal{V}_n, \mathcal{L}_n)$. The law of J_n is a priori random on the probability space of the environment. We assume that J_n is reversible and admits a unique invariant measure π_n.

The process we are interested in, X_n, is defined as a time change of J_n. To this end we set

$$\lambda_n(x) \equiv C\pi_n(x)/\tau_n(x), \tag{1.1}$$

where $C > 0$ is a model dependent constant, and define the clock process

$$\widetilde{S}_n(k) = \sum_{i=0}^{k-1} \lambda_n^{-1}(J_n(i))e_{n,i}, \quad k \in \mathbb{N}\ , \tag{1.2}$$

where $\{e_{n,i} : i \in \mathbb{N}_0, n \in \mathbb{N}\}$ is an i.i.d. array of mean 1 exponential random variables, independent of J_n and the random environment. The continuous time process X_n is then given by

$$X_n(t) = J_n(k), \quad \text{if } \widetilde{S}_n(k) \leq t < \widetilde{S}_n(k+1) \quad \text{for some } k \in \mathbb{N},\ t > 0\ . \tag{1.3}$$

One verifies readily that X_n is a continuous time Markov jump process with infinitesimal generator

$$\lambda_n(x,y) \equiv \lambda_n(x) p_n(x,y), \tag{1.4}$$

and invariant measure that assigns to $x \in \mathcal{V}_n$ the mass $\tau_n(x)$.

To fix notation we denote by $\mathcal{F}^J$ and $\mathcal{F}^X$ the σ-algebras generated by the variables J_n and X_n, respectively. We write P_{π_n} for the law of the process J_n, conditional on $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}_{\mu_n}$ the law of X_n conditional on $\mathcal{F}$.

In [9, 8] and [6], the main aim was to find criteria when there are constants, a_n, c_n, satisfying $a_n, c_n \uparrow \infty$, as $n \to \infty$, and such that the process

$$S_n(t) \equiv c_n^{-1} \widetilde{S}_n(\lfloor a_n t \rfloor) = c_n^{-1} \sum_{i=0}^{\lfloor a_n t \rfloor - 1} \lambda_n^{-1}(J_n(i)) e_{n,i}, \quad t > 0, \tag{1.5}$$

converges in a suitable sense to a stable subordinator. The constants c_n are the time scale on which we observe the continuous time Markov process X_n, while a_n is the number of steps the jump chain J_n makes during that time. In order to get convergence to an α-stable subordinator, for $\alpha \in (0,1)$, one typically requires that the λ^{-1}'s observed on the time scales c_n have a regularly varying tail distribution with index $-\alpha$. In this paper we ask when there are constants, a_n, c_n, α_n, satisfying $a_n, c_n \uparrow \infty$ and $\alpha_n \downarrow 0$ respectively, as $n \to \infty$, and such that the process $(S_n)^{\alpha_n}$ converges in a suitable sense to an extremal process.

1.2. **Main Theorems.** We now state three theorems, beginning with an abstract one that we next specialize to the setting of Section 1.1. Specifically, consider a triangular array of positive random variables, $Z_{n,i}$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let α_n and a_n be sequences such that $\alpha_n \downarrow 0$ and $a_n \uparrow \infty$ as $n \to \infty$, respectively. Our first theorem gives conditions that ensure that the sequence of processes $(S_n)^{\alpha_n}$, where $S_n(0) = 0$ and

$$S_n(t) \equiv \sum_{i=1}^{\lfloor a_n t \rfloor} Z_{n,i}, \quad t > 0, \tag{1.6}$$

converges to an extremal process. Recall that an extremal process, M, is a continuous time process whose finite-dimensional distributions are given as follows: for any $k \in \mathbb{N}$, $t_1, \dots, t_k > 0$, and $x_1 \leq \dots \leq x_k \in \mathbb{R}$,

$$P\left(M(t_1) \leq x_1, \dots, M(t_k) \leq x_k\right) = F^{t_1}(x_1) F^{t_2 - t_1}(x_2) \cdots F^{t_k - t_{k-1}}(x_k), \tag{1.7}$$

where F is a distribution function on $\mathbb{R}$.

Theorem 1.1. *Let ν be a sigma-finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\nu(0,\infty) = \infty$. Assume that there exist sequences a_n, α_n such that for all continuity points x of the distribution function of ν, for all $t > 0$, in $\mathcal{P}$-probability,*

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor a_n t \rfloor} \mathcal{P}\left(Z_{n,i}^{\alpha_n} > x \middle| \mathcal{F}_{n,i-1}\right) = t\nu(x,\infty), \tag{1.8}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor a_n t\rfloor} \left[\mathcal{P}\left(Z_{n,i}^{\alpha_n} > x|\mathcal{F}_{n,i-1}\right)\right]^2 = 0, \tag{1.9}$$

where $\mathcal{F}_{n,i}$ denotes the σ-algebra generated by the random variables $Z_{n,j}, j \le i$. If, moreover, for all $t > 0$

$$\limsup_{n\to\infty} \left(\sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}\mathbb{1}_{Z_{n,i}\le \delta^{1/\alpha_n}} \delta^{-1/\alpha_n} Z_{n,i} \right)^{\alpha_n} < \infty, \quad \forall\, \delta > 0\ , \tag{1.10}$$

then, as $n \to \infty$,

$$(S_n)^{\alpha_n} \stackrel{J_1}{\Longrightarrow} M_\nu, \tag{1.11}$$

where M_ν is an extremal process with one-dimensional distribution function $F(x) = \exp(-\nu(x,\infty))$. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

In the sequel we denote by $\stackrel{J_1}{\Longrightarrow}$ weak convergence in $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

In order to use Theorem 1.1 in the Markov jump process setting of Section 1.1, we specify $Z_{n,i}$. In doing this we will be guided by the knowledge acquired in earlier works [9, 8, 6]: introducing a new scale θ_n we take $Z_{n,i}$ to be a block sum of length θ_n, i.e. we set

$$Z_{n,i} \equiv \sum_{j=(i-1)\theta_n+1}^{i\theta_n} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}\ . \tag{1.12}$$

The rôle of θ_n is to de-correlate the variables $Z_{n,i}$ under the law $\mathcal{P}_{\mu_n}$. In models with uncorrelated environments and where the probability of revisiting points is small, one may hope to take $\theta_n = 1$. When the environment is correlated and the chain J_n is rapidly mixing, one may try to choose $\theta_n \ll a_n$ in such a way that, the variables $Z_{n,i}$ are close to independent. These two situations were encountered in the random hopping dynamics of the Random Energy Model in [8], and the p-spin models in [6] respectively. Theorem 1.2 below specializes Theorem 1.1 to these $Z_{n,i}$'s.

For $y \in \mathcal{V}_n$ and $u > 0$ let

$$Q_n^u(y) \equiv \mathcal{P}_y\left(\sum_{j=1}^{\theta_n} \lambda_n^{-1}(J_n(j))e_{n,j} > c_n u^{1/\alpha_n}\right) \tag{1.13}$$

be the tail distribution of the blocked jumps of X_n, when X_n starts in y. Furthermore, for $k_n(t) \equiv \lfloor\lfloor a_n t\rfloor/\theta_n\rfloor$, $t > 0$, and $u > 0$ define

$$\nu_n^{J,t}(u,\infty) \equiv \sum_{i=1}^{k_n(t)} \sum_{y\in\mathcal{V}_n} p_n(J_n(\theta_n i), y)Q_n^u(y)\ , \tag{1.14}$$

$$(\sigma_n^{J,t})^2(u,\infty) \equiv \sum_{i=1}^{k_n(t)} \left[\sum_{y\in\mathcal{V}_n} p_n(J_n(\theta_n i), y)Q_n^u(y)\right]^2 . \tag{1.15}$$

Using this notation, we rewrite Conditions (1.8)-(1.10). Note that $Q_n^u(y)$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and so are the quantities $\nu_n^{J,t}(u,\infty)$ and $\sigma_n^{J,t}(u,\infty)$. The conditions below are stated for fixed realization of the random environment as well as for given sequences a_n, c_n, θ_n, and α_n such that $a_n, c_n \uparrow \infty$, and $\alpha_n \downarrow 0$ as

$n \to \infty$.
Condition (1) Let ν be a σ-finite measure on $(0, \infty)$ with $\nu(0, \infty) = \infty$ and such that for all $t > 0$ and all $u > 0$

$$\lim_{n\to\infty} P_{\mu_n}\left(\left|\nu_n^{J,t}(u,\infty) - t\nu(u,\infty)\right| > \varepsilon\right) = 0 \ , \quad \forall\varepsilon > 0 \ . \tag{1.16}$$

Condition (2) For all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} P_{\mu_n}\left((\sigma_n^{J,t})^2(u,\infty) > \varepsilon\right) = 0 \ , \quad \forall\varepsilon > 0 \ . \tag{1.17}$$

Condition (3) For all $t > 0$ and all $\delta > 0$

$$\limsup_{n\to\infty} \left(\sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}_{\mu_n} \mathbb{1}_{\{\lambda_n^{-1}(J_n(i))e_{n,i} \leq \delta^{1/\alpha_n} c_n\}} (c_n\delta^{1/\alpha_n})^{-1}\lambda_n^{-1}(J_n(i))e_{n,i}\right)^{\alpha_n} < \infty \ . \tag{1.18}$$

Condition (0) For all $v > 0$,

$$\lim_{n\to\infty} \sum_{x\in\mathcal{V}_n} \mu_n(x) e^{-v^{1/\alpha_n} c_n \lambda_n(x)} = 0 \ . \tag{1.19}$$

For $t > 0$ set

$$\left(S_n^b(t)\right)^{\alpha_n} \equiv \left(\sum_{i=1}^{k_n(t)} \left(\sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}\right) + c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}\right)^{\alpha_n} . \tag{1.20}$$

Theorem 1.2. *If for a given initial distribution μ_n and given sequences a_n, c_n, θ_n, and α_n, Conditions (0)-(3) are satisfied $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability, then*

$$\left(S_n^b\right)^{\alpha_n} \stackrel{J_1}{\Longrightarrow} M_\nu \ , \tag{1.21}$$

where convergence holds $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability.

Remark. Theorem 1.2 tells us that the blocked clock process $(S_n^b)^{\alpha_n}$converges to M_ν weakly in $D([0,\infty))$ equipped with the Skorokhod J_1-topology. This implies that the clock process $(S_n)^{\alpha_n}$ converges to the same limit in the weaker M_1-topology (see [6] for further discussion).

Remark. The extra Condition (0) serves to guarantee that the last term in (1.20) is asymptotically negligible.

Finally, following [6], we specialize Conditions (1)-(3) under the assumption that the chain J_n obeys a mixing condition (see Condition (2-1) below). Conditions (1)-(2) of Theorem 1.2 are then reduced to laws of large numbers for the random variables $Q_n^u(y)$. Again we state these conditions for fixed realization of the random environment and given sequences a_n, c_n, θ_n, and α_n.
Condition (1-1) Let J_n be a periodic Markov chain with period q. There exists a positive decreasing sequence ρ_n, satisfying $\rho_n \downarrow 0$ as $n \to \infty$, such that, for all pairs $x, y \in \mathcal{V}_n$, and all $i \geq 0$,

$$\sum_{k=0}^{q-1} P_{\pi_n}\left(J_n(i+\theta_n+k) = y, J_n(0) = x\right) \leq (1+\rho_n)\pi_n(x)\pi_n(y) \ . \tag{1.22}$$

Condition (2-1) There exists a σ-finite measure ν with $\nu(0,\infty) = \infty$ and such that

$$\nu_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \pi_n(x) Q_n^u(x) \to t\nu(u,\infty) \ , \tag{1.23}$$

and

$$(\sigma_n^t)^2(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \sum_{x'\in\mathcal{V}_n} \pi_n(x) p_n^{(2)}(x,x') Q_n^u(x) Q_n^u(x') \to 0\,, \tag{1.24}$$

where $p_n^{(2)}(x,x') = \sum_{y\in\mathcal{V}_n} p_n(x,y)p_n(y,x')$ are the 2-step transition probabilities.

Condition (3-1) For all $t>0$ and $\delta>0$

$$\limsup_{n\to\infty} \left(\lfloor a_n t\rfloor \mathcal{E}_{\pi_n} \mathbb{1}_{\{\lambda_n^{-1}(J_n(1))e_{n,1}\le c_n\delta^{1/\alpha_n}\}} c_n^{-1}\delta^{-1/\alpha_n}\lambda_n^{-1}(J_n(1))e_{n,1}\right)^{\alpha_n} < \infty\,. \tag{1.25}$$

Theorem 1.3. *Let $\mu_n=\pi_n$. If for given sequences $a_n, c_n, \theta_n \ll a_n$, and α_n, Conditions (1-1)-(3-1) and (0) are satisfied $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability, then $(S_n^b)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$, $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability.*

1.3. **Application to the p-spin SK model.** In this section we illustrate the power of Theorem 1.3 by applying it to the p-spin SK models, including the SK model itself, i.e. $p\ge 2$. The underlying graph $\mathcal{V}_n$ is the hypercube $\Sigma_n=\{-1,1\}^n$. The Hamiltonian of the p-spin SK model is a Gaussian process, H_n, on Σ_n with zero mean and covariance

$$\mathbb{E}H_n(x)H_n(x') = nR_n(x,x')^p, \tag{1.26}$$

where $R_n(x,x') \equiv 1-\frac{2\,\mathrm{dist}(x,x')}{n}$ and $\mathrm{dist}(\cdot,\cdot)$ is the graph distance on Σ_n,

$$\mathrm{dist}(x,x') \equiv \frac{1}{2}\sum_{i=1}^n |x_i-x_i'|. \tag{1.27}$$

The random environment, $\tau_n(x)$, is defined in terms of H_n through

$$\tau_n(x) \equiv \exp(\beta H_n(x)), \tag{1.28}$$

where $\beta>0$ is the inverse temperature. The Markov chain, J_n, is chosen as the simple random walk on Σ_n, i.e.

$$p_n(x,x') = \begin{cases} \frac{1}{n}, & \text{if } \mathrm{dist}(x,x')=1,\\ 0, & \text{else.}\end{cases} \tag{1.29}$$

This chain has unique invariant measure $\pi_n(x)=2^{-n}$. Finally, choosing $C=2^n$ in (1.1), the mean holding times, $\lambda_n^{-1}(x)$, reduce to $\lambda_n^{-1}(x)=\tau_n(x)$. This defines the so-called *random hopping dynamics*.

In the theorem below the inverse temperature β is to be chosen as a sequence $(\beta_n)_{n\in\mathbb{N}}$ that either diverges or converges to a strictly positive limit.

Theorem 1.4. *Let ν be given by $\nu(u,\infty)\equiv K_p u^{-1}$ for $u\in(0,\infty)$ and $K_p=2p$. Let γ_n,β_n be such that $\gamma_n=n^{-c}$ for $c\in\left(0,\frac{1}{2}\right)$, $\beta_n\ge\beta_0$ for some $\beta_0>0$, and $\gamma_n\beta_n\le O(1)$. Set $\alpha_n\equiv\gamma_n/\beta_n$. Let $\theta_n=3n^2$ be the block length and define the jump scales a_n and time scales c_n via*

$$a_n \equiv \sqrt{2\pi n}\,\gamma_n^{-1}\,e^{\frac{1}{2}\gamma_n^2 n}, \tag{1.30}$$

$$c_n \equiv e^{\gamma_n\beta_n n}. \tag{1.31}$$

Then $\left(S_n^b\right)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$. Convergence holds $\mathbb{P}$-a.s. for $p>5$ and in $\mathbb{P}$-probability for $p=2,3,4$. For $p=5$ it holds $\mathbb{P}$-a.s. if $c\in\left(0,\frac{1}{4}\right)$ and in $\mathbb{P}$-probability else.

Remark. Theorem 1.4 immediately implies that $(S_n)^{\alpha_n} \overset{M_1}{\Longrightarrow} M_\nu$ on $D([0,\infty))$ equipped with the weaker M_1- topology.

In [2] an analogous result is proven in law with respect to the environment for similar conditions on the sequence γ_n and fixed β.

Let us comment on the conditions on γ_n and β_n in Theorem 1.4. They guarantee that $\alpha_n \downarrow 0$ as $n \to \infty$, and that both sequences a_n and c_n diverge as $n \to \infty$. Note here that different choices of the sequence β_n correspond to different time scales c_n. If $\beta_n \to \beta > 0$, as $n \to \infty$, then c_n is sub-exponential in n, while in the case of diverging β_n, c_n can be as large as exponential in $O(n)$. Finally these conditions guarantee that the rescaled tail distribution of the τ_n's, on time scale c_n, is regularly varying with index $-\alpha_n$.

We use Theorem 1.4 to derive the limiting behavior of the time correlation function $\mathcal{C}_n^\varepsilon(t,s)$ which, for $t > 0$, $s > 0$, and $\varepsilon \in (0,1)$ is given by

$$\mathcal{C}_n^\varepsilon(t,s) \equiv \mathcal{P}_{\pi_n}\left(A_n^\varepsilon(t,s)\right) \ , \tag{1.32}$$

where $A_n^\varepsilon(t,s) \equiv \left\{R_n\left(X_n(t^{1/\alpha_n}c_n), X_n((t+s)^{1/\alpha_n}c_n)\right) \geq 1-\varepsilon\right\}$.

Theorem 1.5. *Under the assumptions of Theorem 1.4,*

$$\lim_{n\to\infty} \mathcal{C}_n^\varepsilon(t,s) = \frac{t}{t+s}, \quad \forall \varepsilon \in (0,1),\ t,s > 0. \tag{1.33}$$

Convergence holds $\mathbb{P}$-a.s. for $p > 5$ and in $\mathbb{P}$-probability for $p = 2,3,4$. For $p = 5$ it holds $\mathbb{P}$-a.s. if $c \in \left(0, \frac{1}{4}\right)$ and in $\mathbb{P}$-probability else.

Theorem 1.5 establishes extremal ageing as defined in [2]. Here, de-correlation takes place on time intervals of the form $[t^{1/\alpha_n}, (t+s)^{1/\alpha_n}]$, while in normal ageing it takes place on time intervals of the form $[t, t+s]$.

The remainder of the paper is organized as follows. We prove the results of Section 1.2 in Section 2. Section 3 is devoted to the proofs of the statements of Section 1.3. Finally, an additional lemma is proven in the Appendix.

2. Proofs of the main Theorems

Now we come to the proofs of the theorems of Section 1.2. The proof of Theorem 1.1 hinges on the property that extremal processes can be constructed from Poisson point processes. Namely, if $\xi' = \sum_{k\in\mathbb{N}} \delta_{\{t'_k, x'_k\}}$ is a Poisson point process on $(0,\infty)\times(0,\infty)$ with mean measure $dt \times d\nu'$, where ν' is a σ-finite measure such that $\nu'(0,\infty) = \infty$, then

$$M(t) \equiv \sup\{x'_k : \ t'_k \leq t\}, \quad t > 0, \tag{2.1}$$

is an extremal process with 1-dimensional marginal

$$F_t(u) = e^{-t\nu'(u,\infty)}. \tag{2.2}$$

(See e.g. [15], Chapter 4.3.). This was used in [7] to derive convergence of maxima of random variables to extremal processes from an underlying Poisson point process convergence. Our proof exploits similar ideas and the key fact that the $1/\alpha_n$-norm converges to the sup norm as $\alpha_n \downarrow 0$.

Proof of Theorem 1.1. Consider the sequence of point processes defined on $(0,\infty)\times(0,\infty)$ through

$$\xi_n \equiv \sum_{k\in\mathbb{N}} \delta_{\left\{k/a_n, Z_{n,k}^{\alpha_n}\right\}}. \tag{2.3}$$

By Theorem 3.1 of [7], Conditions (1.8) and (1.9) immediately imply that $\xi_n \overset{n\to\infty}{\Rightarrow} \xi$, where ξ is a Poisson point process with intensity measure $dt \times d\nu$.

The remainder of the proof can be summarized as follows. In the first step we construct $(S_n(t))^{\alpha_n}$ from ξ_n by taking the α_n^{th} power of the sum over all points $Z_{n,k}$ up to time $\lfloor a_n t \rfloor$. To this end we introduce a truncation threshold δ and split the ordinates of ξ_n into

$$Z_{n,k}^{\alpha_n} = Z_{n,k}^{\alpha_n} \mathbb{1}_{Z_{n,k}^{\alpha_n} \le \delta} + Z_{n,k}^{\alpha_n} \mathbb{1}_{Z_{n,k}^{\alpha_n} > \delta}. \tag{2.4}$$

Applying a summation mapping to $Z_{n,k}^{\alpha_n} \mathbb{1}_{Z_{n,k}^{\alpha_n} > \delta}$, we show that the resulting process converges to the supremum mapping of a truncated version of ξ. More precisely, let $\delta > 0$. Denote by $\mathcal{M}_p$ the space of point measures on $(0,\infty) \times (0,\infty)$. For $n \in \mathbb{N}$ let T_n^δ be the functional on $\mathcal{M}_p$, whose value at $m = \sum_{k\in\mathbb{N}} \delta_{\{t_k, j_k\}}$ is

$$(T_n^\delta m)(t) = \left(\sum_{t_k \le t} j_k^{1/\alpha_n} \mathbb{1}_{\{j_k > \delta\}} \right)^{\alpha_n}, \quad t > 0. \tag{2.5}$$

Let T^δ be the functional on $\mathcal{M}_p$ given by

$$(T^\delta m)(t) = \sup \left\{ j_k \mathbb{1}_{\{j_k > \delta\}} : t_k \le t \right\}, \quad t > 0 \; . \tag{2.6}$$

We show that $T_n^\delta \xi_n \overset{J_1}{\Longrightarrow} T^\delta \xi$ as $n \to \infty$.

In the second step we prove that the small terms, as $\delta \to 0$ and $n \to \infty$, do not contribute to $(S_n)^{\alpha_n}$, i.e. that for $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathcal{P}\left(\rho_\infty \left(T_n^\delta \xi_n, S_n^{\alpha_n} \right) > \varepsilon \right) = 0 \; , \tag{2.7}$$

where ρ_∞ denotes the Skorokhod metric on $D([0,\infty))$. Moreover, observe that $T^\delta \xi \overset{J_1}{\Longrightarrow} M$ as $\delta \to 0$. Then, by Theorem 4.2 from [3], the assertion of Theorem 1.1 follows.

Step 1*:* To prove that $T_n^\delta \xi_n \overset{J_1}{\Longrightarrow} T^\delta \xi$ as $n \to \infty$ we use a continuous mapping theorem, namely Theorem 5.5 from [3]. Since the mappings T_n^δ and T^δ are measurable, it is sufficient to show that the set

$$\mathcal{E} = \left\{ m \in \mathcal{M}_p : \exists \; (m_n)_{n\in\mathbb{N}} \text{ s.t. } m_n \overset{v}{\to} m, \text{ but } T_n^\delta m_n \overset{J_1}{\not\Longrightarrow} T^\delta m \right\}, \tag{2.8}$$

where $\overset{v}{\to}$ denotes vague convergence in $\mathcal{M}_p$, is a null set with respect to the distribution of ξ. For the Poisson point process ξ it is enough to show that $\mathcal{P}_\xi\left(\mathcal{E}^c \cap \mathcal{D}\right) = 1$, where

$$\mathcal{D} \equiv \left\{ m \in \mathcal{M}_p : m\left((0,t] \times [j,\infty)\right) < \infty \; \forall t, j > 0 \right\}. \tag{2.9}$$

Let $\mathcal{C}_{T^\delta} \equiv \left\{ t > 0 : \; \mathcal{P}_\xi \left(\left\{ m : \; T^\delta m\,(t) = T^\delta m\,(t-) \right\} \right) = 1 \right\}$ be the set of continuity points of ξ. By definition of the Skorokhod metric, we consider $m \in \mathcal{D}$, $a, b \in \mathcal{C}_{T^\delta}$, and $(m_n)_{n\in\mathbb{N}}$ such that $m_n \overset{v}{\to} m$ and show that

$$\lim_{n\to\infty} \rho_{[a,b]} \left(T_n^\delta m_n, T^\delta m \right) = 0 \; , \tag{2.10}$$

where $\rho_{[a,b]}$ denotes the Skorokhod metric on $[a,b]$. Since $m \in \mathcal{D}$, there exist continuity points x, y of m such that $m((a,b) \times (\delta,\infty)) = m((a,b) \times (x,y)) < \infty$. Then, Lemma 2.1 from [13] yields that m_n also has this property for large enough n. Moreover, the points of m_n in $(a,b) \times (x,y)$ converge to the ones of m (cf. Lemma I.14 in [14]). Finally, we use that $\alpha_n \downarrow 0$ as $n \to \infty$ and thus T_n^δ can be viewed as the $1/\alpha_n$-norm, which converges as $n \to \infty$ to the sup-norm T^δ. Therefore, $T_n^\delta \xi_n \overset{J_1}{\Longrightarrow} T^\delta \xi$ as $n \to \infty$.

Step 2*:* We prove (2.7) by showing that the assertion holds true for the Skorokhod metric on $D([0,k])$ for every $k \in \mathbb{N}$. Assume without loss of generality that $k = 1$. Let $\varepsilon > 0$. We have that

$$\begin{aligned} &\mathcal{P} \quad \left(\sup_{0\leq t\leq 1} \left| T_n^\delta \xi_n\left(t\right) - S_n^{\alpha_n}\left(t\right)\right| > \varepsilon\right) \\ &= \quad \mathcal{P}\left(\sup_{0\leq t\leq 1}\left|\left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}>\delta^{1/\alpha_n}}\right)^{\alpha_n} - \left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\right)^{\alpha_n}\right| > \varepsilon\right) . \end{aligned} \tag{2.11}$$

Since for n large enough $\alpha_n < 1$, we know by Jensen inequality that

$$\left|\left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}>\delta^{1/\alpha_n}}\right)^{\alpha_n} - \left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\right)^{\alpha_n}\right| \leq \left|\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}\leq\delta^{1/\alpha_n}}\right|^{\alpha_n} , \tag{2.12}$$

and therefore

$$(2.11) \leq \mathcal{P}\left(\sup_{0\leq t\leq 1}\left|\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}\leq\delta^{1/\alpha_n}}\right|^{\alpha_n} > \varepsilon\right) . \tag{2.13}$$

All summands are non-negative. Hence the supremum is attained for $t = 1$. Applying a first order Chebychev and Jensen inequality, we obtain that (2.13) is bounded above by

$$\varepsilon^{-1}\left(\sum_{i=1}^{a_n} \mathcal{E}\mathbb{1}_{Z_{n,i}\leq\delta^{1/\alpha_n}} Z_{n,i}\right)^{\alpha_n} = \frac{\delta}{\varepsilon}\left(\sum_{i=1}^{a_n} \mathcal{E}\mathbb{1}_{Z_{n,i}\leq\delta^{1/\alpha_n}}\delta^{-1/\alpha_n} Z_{n,i}\right)^{\alpha_n} . \tag{2.14}$$

By (1.10) the sum is bounded in n and hence, as $\delta \to 0$, (2.14) tends to zero. This concludes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Throughout we fix a realisation $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^b(t) \equiv S_n^b(t) - c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}, \qquad t > 0. \tag{2.15}$$

$(S_n^b(t))^{\alpha_n}$ differs from $(\widehat{S}_n^b(t))^{\alpha_n}$ by one term. All terms in $(S_n^b(t))^{\alpha_n}$ are non-negative and therefore we conclude by Jensen inequality that, for n large enough,

$$\widehat{S}_n^b(t)^{\alpha_n} \leq S_n^b(t)^{\alpha_n} \leq \widehat{S}_n^b(t)^{\alpha_n} + \left(c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}\right)^{\alpha_n} . \tag{2.16}$$

By Condition (0) the contribution of the term $(c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0})^{\alpha_n}$ is negligible. Thus we must show that under Conditions (1)-(3), $(\widehat{S}_n^b)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$. Recall that $k_n(t) \equiv \lfloor\lfloor a_n t\rfloor/\theta_n\rfloor$ and that for $i \geq 1$,

$$Z_{n,i} \equiv \sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}. \tag{2.17}$$

We apply Theorem 1.1 to the $Z_{n,i}$'s. It is shown in the proof of Theorem 1.2 in [6] that Conditions (1) and (2) imply (1.8) and (1.9). It remains to prove that Condition (3) yields (1.10). Note that for all $i \geq 1$ and all $(i-1)\theta_n + 1 \leq j \leq i\theta_n$,

$$\mathbb{1}_{\{\sum_{j=(i-1)\theta_n+1}^{i\theta_n}\lambda_n^{-1}(J_n(j))e_{n,j}\leq c_n\delta^{1/\alpha_n}\}} \leq \mathbb{1}_{\{\lambda_n^{-1}(J_n(j))e_{n,j}\leq c_n\delta^{1/\alpha_n}\}} . \tag{2.18}$$

Using (2.18), we observe that (1.10) is in particular satisfied if for all $\delta > 0$ and $t > 0$

$$\limsup_{n\to\infty}\left(\sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}_{\mu_n}\mathbb{1}_{\{\lambda_n^{-1}(J_n(j))e_{n,j}\leq c_n\delta^{1/\alpha_n}\}}\delta^{-1/\alpha_n}c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}\right)^{\alpha_n} < \infty , \tag{2.19}$$

which is nothing but Condition (3). This concludes the proof of Theorem 1.2. $\square$

Finally, having Theorem 1.2 and the results from [6], Theorem 1.3 is deduced readily.

Proof of Theorem 1.3. Let μ_n be the invariant measure π_n of the jump chain J_n. By Proposition 2.1 of [6] we know that Conditions (0), (1-1), and (2-1) imply Conditions (0)-(2) of Theorem 1.2. Moreover, since $\mu_n = \pi_n$, Condition (3-1) is Condition (3). Thus, the conditions of Theorem 1.2 are satisfied under the assumptions of Theorem 1.3 and this yields the claim. □

3. Application to the p spin SK model

This section is devoted to the proof of Theorem 1.4. We show that the conditions of Theorem 1.3 are satisfied for the particular choices of the sequences a_n, c_n, θ_n, and α_n.

The following lemma from [8] (Proposition 3.1) implies that Condition (1-1) holds true for $\theta_n = 3n^2$.

Lemma 3.1. *Let P_{π_n} be the law of the simple random walk on Σ_n started in the uniform distribution. Let $\theta_n = 3n^2$. Then, for any $x, y \in \Sigma_n$, and any $i \geq 0$,*

$$\left| \sum_{k=0}^{1} P_{\pi_n}\left(J_n(\theta_n + i + k) = y, J_n(0) = x\right) - 2\pi_n(x)\pi_n(y) \right| \leq 2^{-3n+1}. \tag{3.1}$$

The proof of Condition (2-1) comes in three parts. We first show that $\mathbb{E}\nu_n^t(u,\infty)$ converges to $t\nu(u,\infty)$. Next we prove that $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability, the limit of $\nu_n^t(u,\infty)$ concentrates for all $u > 0$ and all $t > 0$ around its expectation. Lastly we verify that the second part of Condition (2-1) is satisfied in the same convergence mode with respect to the random environment.

3.1. Convergence of $\mathbb{E}\nu_n^t(u,\infty)$.

Proposition 3.2. *For all $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} \mathbb{E}\nu_n^t(u,\infty) = \nu^t(u,\infty) \equiv K_p t u^{-1} \ . \tag{3.2}$$

The proof of Proposition 3.2 centers on the following key proposition.

Proposition 3.3. *Let for $t > 0$ and an arbitrary sequence u_n,*

$$\bar{\nu}_n^t(u_n,\infty) = k_n(t)\, \mathcal{P}_{\pi_n}\left(\max_{i=1,\dots,\theta_n} \lambda_n^{-1}(J_n(i))e_{n,i} > u_n^{1/\alpha_n} c_n \right) . \tag{3.3}$$

Then, for all $u > 0$ and $t > 0$,

$$\lim_{n\to\infty} \mathbb{E}\ \bar{\nu}_n^t(u,\infty) = \nu^t(u,\infty) \ . \tag{3.4}$$

The same holds true when u is replaced by $u_n = u\ \theta_n^{-\alpha_n}$.

Proof of Proposition 3.2. By definition, $\nu_n^t(u,\infty)$ is given by

$$\nu_n^t(u,\infty) = k_n(t)\, \mathcal{P}_{\pi_n}\Big(\sum_{i=1}^{\theta_n} \lambda_n^{-1}(J_n(i))e_{n,i} > u^{1/\alpha_n} c_n \Big). \tag{3.5}$$

The assertion of Proposition 3.2 is then deduced from Proposition 3.3 using the upper and lower bounds

$$\bar{\nu}_n^t(u,\infty) \leq \nu_n^t(u,\infty) \leq \bar{\nu}_n^t(u\theta_n^{-\alpha_n},\infty) \ . \tag{3.6}$$

□

The proof of Proposition 3.3, which is postponed to the end of this section, relies on three Lemmata. In Lemma 3.4 we show that (3.4) holds true if we replace the underlying Gaussian process by a simpler Gaussian process H^1. Lemma 3.5 yields (3.4) for the maximum over a properly chosen random subset of indices of H^1. We use Lemma 3.7 to conclude the proof of Proposition 3.3.

We start by introducing the Gaussian process H^1. Let v_n be a sequence of integers, where each member is of order n^ω for $\omega \in \left(c + \frac{1}{2}, 1\right)$. Then, H^1 is a centered Gaussian process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance structure

$$\Delta^1_{i,j} = \begin{cases} 1 - 2pn^{-1}|i-j|, & \text{if } \lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor, \\ 0, & \text{else.} \end{cases} \tag{3.7}$$

For a given process $U = \{U_i,\ i \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and an index set I define

$$F_n(u_n, U, I) \equiv \mathbb{P}\left(\max_{i \in I} e^{\sqrt{n}\beta_n U_i} > u_n^{1/\alpha_n} c_n\right), \tag{3.8}$$

and for a process $\widetilde{U} = \{\tilde{U}_i,\ i \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ that may also be dependent on $\mathcal{F}^J$

$$G_n(u_n, \widetilde{U}, I) \equiv \mathcal{P}_{\pi_n}\left(\max_{i \in I} e^{\sqrt{n}\beta_n \widetilde{U}_i} e_{n,i} > u_n^{1/\alpha_n} c_n \,\middle|\, \mathcal{F}^J\right). \tag{3.9}$$

Lemma 3.4. *For all $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} k_n(t)\mathbb{E}G_n(u, H^1, [\theta_n]) = \nu^t(u, \infty), \tag{3.10}$$

where $[k] \equiv \{1, \ldots, k\}$ for $k \in \mathbb{N}$. The same holds true when u is replaced by $u_n = u\,\theta_n^{-\alpha_n}$.

We prove Proposition 3.3 and Lemmata 3.4, 3.5, and 3.7 for fixed $u > 0$ only. To show that the claims also hold for $u_n = u\theta_n^{-\alpha_n}$, it is a simple rerun of their proofs, using $\theta_n^{-\alpha_n} \uparrow 1$ as $n \to \infty$.

Proof. It is shown in Proposition 2.1 of [2] that, by setting the exponentially distributed random variables to 1 in (3.9) and taking expectation with respect to the random environment, we get for all $u > 0$ that

$$\lim_{n\to\infty} a_n v_n^{-1} F_n(u, H^1, [v_n]) = \nu(u, \infty). \tag{3.11}$$

Assume for simplicity that θ_n is a multiple of v_n. Note that blocks of H^1 of length v_n are independent and identically distributed. Thus,

$$\begin{aligned} k_n(t)F_n(u, H^1, [\theta_n]) &= k_n(t)\left(1 - \left(1 - F_n(u, H^1, [v_n])\right)^{\theta_n/v_n}\right) \\ &\sim k_n(t)\theta_n v_n^{-1} F_n(u, H^1, [v_n]) \\ &\overset{n\to\infty}{\longrightarrow} \nu^t(u, \infty). \end{aligned} \tag{3.12}$$

To show that $k_n(t)\mathbb{E}G_n(u, H^1, [\theta_n])$ also converges to $\nu^t(u, \infty)$ as $n \to \infty$ we use same arguments as in (3.12) and prove that $a_n v_n^{-1}\mathbb{E}G_n(u, H^1, [v_n]) \to \nu(u, \infty)$ as $n \to \infty$. Using Fubini we have that

$$\begin{aligned} \frac{a_n}{v_n}\mathbb{E}G_n(u, H^1, [v_n]) &= \frac{a_n}{v_n}\int_{c_n u^{1/\alpha_n}}^{\infty} dz \int_0^\infty dy \frac{f_{\max_{i\in[v_n]} e_{n,i}}(y)}{y} f_{\max_{i\in[v_n]} e^{\beta_n\sqrt{n}H^1(i)}}\left(\tfrac{z}{y}\right) \\ &= \frac{a_n}{v_n}\int_0^\infty dy f_{\max_{i\in[v_n]} e_{n,i}}(y) F_n(u\,y^{-\alpha_n}, H^1, [v_n]), \end{aligned} \tag{3.13}$$

where $f_Z(\cdot)$ denotes the density function of Z. Since we want to use computations from the proof of Proposition 2.1 in [2], it is essential that the integration area over y is bounded from below and above. We bound (3.13) from above by

$$\begin{aligned}
(3.13) \quad &\leq \quad a_n v_n^{-1} \mathcal{P}\Big(\max_{i=1,\dots,v_n} e_{n,i} \leq e^{-n v_n^{-1-\delta}} \Big) && (3.14)\\
&+ \quad a_n v_n^{-1} \int_{e^{-n v_n^{-1-\delta}}}^{e^{n v_n^{-1/2-\delta}}} dy f_{\max_{i\in[v_n]} e_{n,i}}(y) F_n(u\, y^{-\alpha_n}, H^1, [v_n]) && (3.15)\\
&+ \quad a_n v_n^{-1} \mathcal{P}\Big(\max_{i=1,\dots,v_n} e_{n,i} > e^{n v_n^{-1/2-\delta}} \Big) , && (3.16)
\end{aligned}$$

where $\delta > 0$ is chosen in such a way that $n v_n^{-1-\delta}$ diverges and $v_n^{\delta}\gamma_n^2 \downarrow 0$ as $n \to \infty$, i.e. $\delta < \min\left\{2c, \frac{1-\omega}{\omega}\right\}$. Then,

$$(3.14) = a_n v_n^{-1} \Big(1 - \exp\Big(-e^{-n v_n^{-1-\delta}}\Big)\Big)^{v_n} \leq a_n e^{-n v_n^{-\delta}} = o\left(e^{-n v_n^{-\delta}(1-\gamma_n^2 v_n^{\delta})}\right) , \quad (3.17)$$

i.e. (3.14) vanishes as $n \to \infty$. Similarly,

$$(3.16) = a_n v_n^{-1} \Big(1 - \Big(1 - \exp\Big(-e^{-n v_n^{-1/2-\delta}}\Big)\Big)^{v_n}\Big) = o\Big(e^{\gamma_n^2 n - e^{n v_n^{-1/2-\delta}}}\Big) \stackrel{n\to\infty}{\longrightarrow} 0 \,. \quad (3.18)$$

As in equation (2.31) in [2] we see that (3.15) is given by

$$\int_{e^{-n v_n^{-1-\delta}}}^{e^{n v_n^{-1/2-\delta}}} dy \frac{f_{\max_{i\in[v_n]} e_{n,i}}(y)}{\gamma_n^2 v_n} \sum_{k=1}^{v_n} \int_{D_k''} da_2 \cdots da_{v_n} \int_{\log(u y^{-\alpha_n})}^{\infty} da_1 \frac{e^{-h_k(a_1,\dots,a_{v_n})}}{(2\pi)^{\frac{v_n-1}{2}}} , \quad (3.19)$$

where for $k \in \{1, \dots, v_n\}$

$$h_k(a_1, \dots, a_{v_n}) = a_1 - \frac{a_1^2 C_1}{\gamma_n^2 n} - \tfrac{1}{2}\textstyle\sum_{i=2}^{v_n} a_i^2 + \frac{(a_2+\ldots+a_k-a_{k+1}-\ldots-a_{v_n})a_1 C_2}{\gamma_n n} , \quad (3.20)$$

for some constants $C_1, C_2 > 0$ and a sequence of sets $D_k'' \subseteq \mathbb{R}^{v_n-1}$ such that

$$\gamma_n^{-2} v_n^{-1} \sum_{k=1}^{v_n} \int_{D_k''} da_2 \cdots da_{v_n} (2\pi)^{-v_n/2-1/2} e^{-\frac{1}{2}\sum_{i=2}^{v_n} a_i^2} \stackrel{n\to\infty}{\longrightarrow} K_p \,. \quad (3.21)$$

The aim is to separate a_1 from $a_2, \dots, a_{v_n}$ in (3.20). We bound the mixed terms in e^{-h_k} up to an exponentially small error by 1. This can be done using a large deviation argument for $|a_2+\ldots+a_{v_n}|$ together with the fact that $|\log y| \in \left[n v_n^{-1-\delta}, n v_n^{-1/2-\delta}\right]$. Computations yield together with the bounds in (3.19)-(3.21) that, up to a multiplicative error that tends to 1 as $n \to \infty$ exponentially fast, (3.15) is bounded from above by

$$\int_{e^{-n v_n^{-1-\delta}}}^{\infty} dy f_{\max_{i\in[v_n]} e_{n,i}}(y) y^{\alpha_n}\, u^{-1} K_p \leq \nu(u,\infty) \int_0^{\infty} dy f_{\max_{i\in[v_n]} e_{n,i}}(y) y^{\alpha_n} \,. \quad (3.22)$$

Moreover by Jensen inequality,

$$\begin{aligned}
(3.22) \quad &\leq \quad \nu(u,\infty)\Big(\mathcal{E}_{\pi_n} \max_{i\in[v_n]} e_{n,i}\Big)^{\alpha_n}\\
&= \quad \nu(u,\infty)\Big(\int_0^{\infty} dy\, \mathcal{P}\Big(\max_{i\in[v_n]} e_{n,i} > y\Big)\Big)^{\alpha_n}\\
&= \quad \nu(u,\infty)\Big(\int_0^{\infty} dy \Big(1 - \big(1-e^{-y}\big)^{v_n}\Big)\Big)^{\alpha_n}\\
&\leq \quad \nu(u,\infty) v_n^{\alpha_n} , && (3.23)
\end{aligned}$$

which, as $n \to \infty$, converges to $\nu(u, \infty)$.

To conclude the proof of (3.10), we bound (3.13) from below by

$$(3.13) \geq \frac{a_n}{v_n} \int_0^\infty dy f_{e_{n,1}}(y) F_n(u\, y^{-\alpha_n}, H^1, [v_n]) \; . \tag{3.24}$$

To show that the right hand side of (3.24) is greater than or equal to $\nu(u,\infty)$, one proceeds as before. □

In the following we form a random subset of $[\theta_n]$ in such a way that on the one hand, with high probability, it contains the maximum of $e^{\beta_n \sqrt{n} H^1(i)}$ over all $i \in [\theta_n]$. On the other hand it should be a sparse enough subset of $[\theta_n]$ so that we are able to de-correlate the random landscape and deal with the SK model. This dilution idea is taken from [2].

If the maximum of $e^{\beta_n \sqrt{n} H^1(i)}$ crosses the level $c_n u^{1/\alpha_n}$, then it will typically be much larger than $c_n u^{1/\alpha_n}$ so that, due to strong correlation, at least γ_n^{-2} of its direct neighbors will be above the same level. To see this, we consider Laplace transforms. Set for $v > 0$

$$\widehat{F}_n(v, H^1, \theta_n) \equiv \int_0^\infty dz\; e^{-zv} \mathbb{P}\left(\delta_n \sum_{i=1}^{\theta_n} \mathbb{1}_{e^{\beta\sqrt{n} H^1(i)} > c_n u^{1/\alpha_n}} > z\right) \; , \tag{3.25}$$

where $\delta_n \in [0,1]$ for every $n \in \mathbb{N}$. We have that

$$\begin{aligned} \widehat{F}_n(v, H^1, \theta_n) \;&=\; \tfrac{1}{v}\left(1 - \mathbb{E}\exp\left(-\delta_n \sum_{i=1}^{\theta_n} \mathbb{1}_{e^{\beta_n\sqrt{n} H^1(i)} > c_n u^{1/\alpha_n}}\right)\right) \\ &=\; \tfrac{1}{v}\left(1 - \left(\mathbb{E}\exp\left(-\delta_n \sum_{i=1}^{v_n} \mathbb{1}_{e^{\beta_n\sqrt{n} H^1(i)} > c_n u^{1/\alpha_n}}\right)\right)^{\theta_n/v_n}\right). \end{aligned} \tag{3.26}$$

From [2], Proposition 1.3, we deduce that for the choice $\delta_n = \gamma_n^2 \rho_n$, where ρ_n is any diverging sequence of order $O(\log n)$,

$$\lim_{n\to\infty} a_n v_n^{-1}\left(1 - \mathbb{E}\exp\left(-\delta_n \sum_{i=1}^{v_n} \mathbb{1}_{e^{\beta_n\sqrt{n} H^1(i)} > c_n u^{1/\alpha_n}}\right)\right) = \nu(u,\infty) \; . \tag{3.27}$$

Therefore we have for the same choice of δ_n that

$$k_n(t) \widehat{F}_n(v, H^1, \theta_n) \to t v^{-1} \nu(u,\infty) \; . \tag{3.28}$$

From this we conclude that if the maximum is above the level $c_n u^{1/\alpha_n}$ then immediately $O(\gamma_n^{-2})$ are above this level. More precisely, we obtain

Lemma 3.5. *Let ρ_n be as described above. Let $\{\xi_{n,i} : \; i \in \mathbb{N},\; n \in \mathbb{N}\}$ be an array of row-wise independent and identically distributed Bernoulli random variables such that $\mathbb{P}(\xi_{n,i} = 1) = 1 - \mathbb{P}(\xi_{n,i} = 0) = \gamma_n^2 \rho_n$, and such that $\{\xi_{n,i} : i \in \mathbb{N},\; n \in \mathbb{N}\}$ is independent of everything else. Set*

$$\mathcal{I}_k = \{i \in \{1, \ldots, k\} : \; \xi_{n,i} = 1\} \; . \tag{3.29}$$

Then, for all $u > 0$ and $t > 0$

$$\lim_{n\to\infty} k_n(t) \mathbb{E} G_n(u, H^1, \mathcal{I}_{\theta_n}) = \nu^t(u,\infty) \; . \tag{3.30}$$

The same holds true when u is replaced by $u_n = u\; \theta_n^{-\alpha_n}$.

Proof. It is shown in Lemma 2.3 of [2] that

$$\lim_{n\to\infty} a_n v_n^{-1} F_n(u, H^1, \mathcal{I}_{v_n}) = \nu(u,\infty) \; . \tag{3.31}$$

Since the random variables $\xi_{n,i}$ are independent, the claim of Lemma 3.5 is deduced by the same arguments as in (3.12). □

To conclude the proof of Proposition 3.3, we use a Gaussian comparison result. The following lemma is an adaptation of Theorem 4.2.1of [11].

Lemma 3.6. *Let H^0 and H^1 be Gaussian processes with mean 0 and covariance matrix $\Delta^0 = (\Delta^0_{ij})$ and $\Delta^1 = (\Delta^1_{ij})$, respectively. Set $\Delta^m \equiv \left(\Delta^m_{ij}\right) = \left(\max\{\Delta^0_{ij}, \Delta^1_{ij}\}\right)$ and $\Delta^h \equiv h\Delta^0 + (1-h)\Delta^1$, for $h \in [0,1]$. Then, for $s \in \mathbb{R}$,*

$$\begin{aligned} & \mathbb{P}(\max_{i\in I} H^0(i) \le s) - \mathbb{P}(\max_{i\in I} H^1(i) \le s) \\ \le\ & \textstyle\sum_{i,j\in I} (\Delta^0_{ij} - \Delta^1_{ij})^+ \exp\left(-\frac{s^2}{1+\Delta^m_{ij}}\right) \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}}\ , \end{aligned} \tag{3.32}$$

where $(x)^+ \equiv \max\{0, x\}$.

We use Lemma 3.6 to prove that

Lemma 3.7. *Let H^0 be given by $H^0(i) \equiv n^{-1/2} H_n(J_n(i))$, $i \in \mathbb{N}$. For all $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} k_n(t) E_{\pi_n} |\mathbb{E}G_n(u, H^0, \theta_n) - \mathbb{E}G_n(u, H^1, \theta_n)| = 0. \tag{3.33}$$

The same holds true when u is replaced by $u_n = u\theta_n^{-\alpha}$.

Proof. The proof is in the same spirit as that of Proposition 3.1 in [2]. Together with Lemma 3.5, it is sufficient to show that

$$k_n(t) E_{\pi_n} (\mathbb{E}G_n(u, H^1, [\theta_n]) - \mathbb{E}G_n(u, H^0, [\theta_n]))^+ \to 0 \tag{3.34}$$

and

$$k_n(t) E_{\pi_n} |\mathbb{E}G_n(u, H^1, \mathcal{I}_{\theta_n}) - \mathbb{E}G_n(u, H^0, \mathcal{I}_{\theta_n})| \to 0\ . \tag{3.35}$$

We do this by an application of Lemma 3.6. Let $\hat{s}_n$ be given by

$$\hat{s}_n = \tfrac{1}{\sqrt{n}\beta_n} \left(\log c_n + \tfrac{\beta_n}{\gamma_n} \log u - \max_{i\in[\theta_n]} \log e_{n,i}\right)\ . \tag{3.36}$$

Then we obtain by Lemma 3.6 that

$$\begin{aligned} & (3.34) \\ =\ & k_n(t) E_{\pi_n} \left(\mathbb{E}\mathcal{E}_{\pi_n}\left[\mathbb{1}_{\max_{i\in[\theta_n]} H^1(i)\le \hat{s}_n} - \mathbb{1}_{\max_{i\in[\theta_n]} H^0(i)\le \hat{s}_n} \mid \mathcal{F}^J\right]\right)^+ \\ \le\ & k_n(t) E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} (\Delta^1_{ij} - \Delta^0_{ij})^+ \mathcal{E}_{\pi_n} e^{-\hat{s}_n^2 (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}}\ . \end{aligned} \tag{3.37}$$

To remove the exponentially distributed random variables $e_{n,i}$ in (3.37), let $B_n = \{1 \le \max_{i\in[\theta_n]} e_i \le n\}$. We have for $s_n = (n^{1/2}\beta_n)^{-1}\left(\log c_n + \frac{\beta_n}{\gamma_n}\log u - \log n\right)$ that

$$\mathcal{E}_{\pi_n}\left(\mathbb{1}_{B_n} \exp\left(-\hat{s}_n^2 (1+\Delta^m_{ij})^{-1}\right)\right) \le \exp\left(-s_n^2(1+\Delta^m_{ij})^{-1}\right)\ . \tag{3.38}$$

One can check that $k_n(t)\mathcal{P}(B_n^c) \downarrow 0$. Moreover, by definition of s_n, there exists for all $u > 0$ a constant $C < \infty$ such that for n large enough

$$(3.34) \le C k_n(t) E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} (\Delta^1_{ij} - \Delta^0_{ij})^+ e^{-\gamma_n^2 n (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}}\ . \tag{3.39}$$

Likewise we deal with (3.35). The terms in (3.35) are non-zero if and only if $i, j \in \mathcal{I}_{\theta_n}$. By assumption, the probability of this event is $(\gamma_n^2\rho_n)^2$. Hence, (3.35) is bounded above by

$$C k_n(t) (\gamma_n^2\rho_n)^2 E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} |\Delta^0_{ij} - \Delta^1_{ij}| e^{-\gamma_n^2 n (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}}\ . \tag{3.40}$$

We divide the summands in (3.39) and (3.40) respectively into two parts: pairs of i, j such that $\lfloor i/v_n \rfloor \ne \lfloor j/v_n \rfloor$ and those such that $\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor$. If $\lfloor i/v_n \rfloor \ne \lfloor j/v_n \rfloor$ then

we have by definition of H^1 that $\Delta^1_{ij}=0$. For i,j such that $\lfloor i/v_n\rfloor=\lfloor j/v_n\rfloor$, we have $\Delta^1_{ij}\le\Delta^0_{ij}$. In view of this, we get after some computations that

$$(3.39)\le Ck_n(t)E_{\pi_n}\left[\sum_{\lfloor i/v_n\rfloor\neq\lfloor j/v_n\rfloor}^{\theta_n}(\Delta^0_{ij})^- e^{-\gamma_n^2 n}\right], \tag{3.41}$$

and

$$\begin{aligned}(3.40) \quad &\le \quad Ck_n(t)\gamma_n^4\rho_n^2 E_{\pi_n}\left[\sum_{\lfloor i/v_n\rfloor\neq\lfloor j/v_n\rfloor}^{\theta_n}|\Delta^0_{ij}|e^{-\gamma_n^2 n(1+\Delta^0_{ij})^{-1}}\right.\\ &+ \quad \left.\sum_{\lfloor i/v_n\rfloor=\lfloor j/v_n\rfloor}^{\theta_n}|\Delta^0_{ij}-\Delta^1_{ij}|e^{-\gamma^2 n(1+\Delta^0_{ij})^{-1}}(1-(\Delta^0_{ij})^2)^{-\frac12}\right]. \end{aligned}\tag{3.42}$$

Since $(\Delta^0_{ij})^-=O(n)$ we know by definition of a_n and θ_n that

$$(3.41)\le C\theta_n n^{3/2}\alpha_n^{-1}e^{-\frac12\gamma_n^2 n}, \tag{3.43}$$

which tends to zero as $n\to\infty$. Thus (3.34) holds true.

To conclude the proof of (3.35) we use Lemma 4.1 from the appendix. We get that (3.40) is bounded above by

$$\bar{C}ta_n\sum_{d=0}^n e^{-\gamma_n^2 n(1+d)^{-1}}\left(\frac{d^2}{v_n n}\mathbb{1}_{d\le v_n}+\frac{\exp(\eta\gamma_n^2\min\{d,n-d\})}{v_n\gamma_n^2}\right), \tag{3.44}$$

for some $\bar{C}<\infty$ and $\eta<\infty$. With the same arguments as in the proof of (3.3) in [2], we obtain that (3.44) tends to zero as $n\to\infty$. □

Proof of Proposition 3.3. Observe that

$$\left|\mathbb{E}\bar\nu_n^t(u,\infty)-\nu^t(u,\infty)\right|=\left|k_n(t)E_{\pi_n}\mathbb{E}G_n(u,H^0,[\theta_n])-\nu^t(u,\infty)\right|, \tag{3.45}$$

which is bounded above by

$$k_n(t)E_{\pi_n}\left|\mathbb{E}G_n(u,H^0,[\theta_n])-\mathbb{E}G_n(u,H^1,[\theta_n])\right|+\left|k_n(t)\mathbb{E}G_n(u,H^1,[\theta_n])-\nu^t(u,\infty)\right|. \tag{3.46}$$

By Lemma 3.4 and Lemma 3.7, both terms vanish as $n\to\infty$ and Proposition 3.3 follows. □

3.2. **Concentration of $\nu_n^t(u,\infty)$.** To verify the first part of Condition (2-1) we control the fluctuation of $\nu_n^t(u,\infty)$ around its mean.

Proposition 3.8. *For all $u>0$ and $t>0$ there exists $C=C(p,t,u)<\infty$, such that*

$$\mathbb{E}\left(\bar\nu_n^t(u,\infty)-\mathbb{E}\bar\nu_n^t(u,\infty)\right)^2\le C\gamma_n^{-2}n^{1-p/2}. \tag{3.47}$$

The same holds true when u is replaced by $u_n=u\theta_n^{-\alpha_n}$. In particular, for $p>5$ and $c\in(0,\frac12)$ or $p=5$ and $c<\frac14$, the first part of Condition (2-1) holds for all $u>0$ and $t>0$, $\mathbb{P}$-a.s.

Proof. Let $\left\{e'_{n,i}:i\in\mathbb{N},n\in\mathbb{N}\right\}$ and J'_n be independent copies of $\{e_{n,i}:i\in\mathbb{N},n\in\mathbb{N}\}$ and J_n respectively. Writing π_n for the initial distribution of J_n and π'_n for that of J'_n, we define

$$\begin{aligned}\bar G_n(u,H^0,[\theta_n]) &\equiv \mathcal{P}_{\pi_n}\left(\max_{i\in[\theta_n]}e^{\beta_n H_n(J_n(i))}e_{n,i}\le c_n u^{1/\alpha_n}\middle|\mathcal{F}^J\right)\\ \bar G_n(u,H^{0'},[\theta_n]) &\equiv \mathcal{P}_{\pi'_n}\left(\max_{i\in[\theta_n]}e^{\beta_n H_n(J'_n(i))}e'_{n,i}\le c_n u^{1/\alpha_n}\middle|\mathcal{F}^{J'}\right).\end{aligned}\tag{3.48}$$

Then, as in (3.21) in [6],

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}_{\pi_n}\bar G_n(u,H^0,[\theta_n])\right)^2 &= \mathbb{E}\mathcal{E}_{\pi_n}\bar G_n(u,H^0,[\theta_n])\mathcal{E}_{\pi'_n}\bar G_n(u,H^{0'},[\theta_n])\\ &= \mathcal{E}_{\pi_n}\mathcal{E}_{\pi'_n}\mathbb{E}\bar G_n(u,V^0,[2\theta_n]),\end{aligned}\tag{3.49}$$

where V^0 is a Gaussian process defined by

$$V^0(i) = \begin{cases} n^{-1/2} H_n(J_n(i)), & \text{if } 1 \le i \le \theta_n, \\ n^{-1/2} H_n(J_n'(i)), & \text{if } \theta_n + 1 \le i \le 2\theta_n \ . \end{cases} \tag{3.50}$$

To further express $\left(\mathbb{E}\mathcal{E}_{\pi_n} \bar{G}_n(u, H^0, [\theta_n])\right)^2$, let V^1 be a centered Gaussian process with covariance matrix

$$\Delta^1_{ij} = \begin{cases} \Delta^0_{ij}, & \text{if } \max\{i,j\} \le \theta_n, \text{ or } \min\{i,j\} \ge \theta_n, \\ 0, & \text{else}, \end{cases} \tag{3.51}$$

where $\Delta^0 = (\Delta^0_{ij})$ denotes the covariance matrix of V^0. Then, as in (3.23) in [6],

$$\left(\mathbb{E}\mathcal{E}_{\pi_n} \bar{G}_n(u, H^0, [\theta_n])\right)^2 = \mathcal{E}_{\pi_n}\mathcal{E}_{\pi_n'} \mathbb{E}\bar{G}_n(u, V^1, [2\theta_n]) \ . \tag{3.52}$$

As in the proof of Lemma 3.7 we use Lemma 3.6 to obtain that

$$\begin{aligned} & k_n^2(t)\mathbb{E}\left(\mathcal{E}_{\pi_n}\bar{G}_n(u, H^0, [\theta_n]) - \mathbb{E}\mathcal{E}_{\pi_n}\bar{G}_n(u, H^0, [\theta_n])\right)^2 \\ \le\ & 2k_n^2(t) \sum_{\substack{1 \le i \le \theta_n \\ \theta_n + 1 \le j \le 2\theta_n}} E_{\pi_n} E_{\pi_n'} \Delta^0_{ij} e^{-\gamma_n^2 n (1+\Delta^0_{ij})^{-1}}. \end{aligned} \tag{3.53}$$

It is shown in (3.29) of [6] that

$$E_{\pi_n} E_{\pi_n'} \mathbb{1}_{\Delta^0_{ij} = \left(\frac{m}{n}\right)^p} = 2^{-n} \binom{n}{(n-m)/2}, \qquad \text{for } m \in \{0, \dots, n\}. \tag{3.54}$$

From this, and with the definition of a_n, we have that

$$\begin{aligned} (3.53) \ \le\ & 2t^2 a_n^2 \sum_{m=0}^{n} 2^{-n} \binom{n}{(n-m)/2} \left(\frac{m}{n}\right)^p \exp\left(-\frac{\gamma_n^2 n}{1 + (\frac{m}{n})^p}\right) \\ \le\ & 2t^2 \gamma_n^{-2} \sum_{m=0}^{n} 2^{-n} n \binom{n}{(n-m)/2} \left(\frac{m}{n}\right)^p \exp\left(\gamma_n^2 n \frac{(\frac{m}{n})^p}{1 + (\frac{m}{n})^p}\right) \\ =\ & 2t^2 \gamma_n^{-2} \sum_{d=0}^{n} 2^{-n} n \binom{n}{d} \left(1 - \frac{2d}{n}\right)^p \exp\left(\gamma_n^2 n \frac{(1 - \frac{2d}{n})^p}{1 + (1 - \frac{2d}{n})^p}\right) \\ \le\ & 2t^2 \gamma_n^{-2} \sum_{d=0}^{n} n^{1/2} \left(1 - \frac{2d}{n}\right)_+^p \exp\left(n \Upsilon_{n,p}\left(\tfrac{d}{n}\right)\right) J_n\left(\tfrac{d}{n}\right) \ , \end{aligned} \tag{3.55}$$

where for $u \in (0,1)$ we set $\Upsilon_{n,p}(u) = \gamma_n^2 - I(u) - \gamma_n^2(1 + |1-2u|^p)^{-1}$ and $J_n(u) = 2^{-n}\binom{n}{\lfloor nu \rfloor}\sqrt{\pi n} e^{nI(u)}$ for $I(u) = u \log u + (1-u)\log(1-u) + \log 2$. Note that (3.55) has the same form as (3.28) in [1]. Following the strategy of [1], we show that there exist $\delta, \delta' > 0$ and $c > 0$ such that

$$\Upsilon_{n,p} \le \begin{cases} -c\left(u - \frac{1}{2}\right)^2, & \text{if } u \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta), \\ -\delta', & \text{else.} \end{cases} \tag{3.56}$$

Since $\gamma_n = n^{-c}$ this can be done, independently of p, as in [2] (cf. (3.19) and (3.20)). Finally, together with the calculations from (3.28) in [1] we obtain that

$$\mathbb{E}\left(\bar{\nu}_n^t(u, \infty) - \mathbb{E}\bar{\nu}_n^t(u,\infty)\right)^2 \le C\gamma_n^{-2} n^{1-p/2}. \tag{3.57}$$

The same arguments and calculations are used to prove that (3.47) also holds when u is replaced by $u_n = u\theta_n^{-\alpha_n}$. Let $p > 5$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c < \frac{1}{4}$. Then, by Borel-Cantelli Lemma, for all $u > 0$ and $t > 0$ there exists a set $\Omega(u,t)$ with $\mathbb{P}(\Omega(u,t)) = 1$ such that on $\Omega(u,t)$, for all $\varepsilon > 0$ and n large enough, we have that $|\bar{\nu}_n^t(u,\infty) - \nu^t(u,\infty)| < \varepsilon$ and $|\bar{\nu}_n^t(u_n,\infty) - \nu^t(u,\infty)| < \varepsilon$. From this we conclude together with (3.6) that, on $\Omega(u,t)$ and for n large enough,

$$\nu^t(u,\infty) - \varepsilon \le \nu_n^t(u,\infty) \le \nu^t(u_n,\infty) + \varepsilon, \tag{3.58}$$

i.e. Condition (2-1) is satisfied, for all $u > 0$ and $t > 0$, $\mathbb{P}$-a.s. $\square$

Proposition 3.9. *Let $p = 2,3,4$ and $c \in (0,\frac{1}{2})$ or $p = 5$ and $c > \frac{1}{4}$. Then, the first part of Condition (2-1) holds in $\mathbb{P}$-probability for all $u > 0$ and $t > 0$.*

Proof. For all $\varepsilon > 0$, we bound $\mathbb{P}\left(|\nu_n^t(u,\infty) - \mathbb{E}(\nu_n^t(u,\infty))| > \varepsilon\right)$ from above by

$$\begin{aligned} & \mathbb{P}\left(|\nu_n^t(u,\infty) - k_n(t)\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})| > \varepsilon/3\right) && (3.59)\\ + \; & \mathbb{P}\left(k_n(t)|\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n}) - \mathbb{E}\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})| > \varepsilon/3\right) && (3.60)\\ + \; & \mathbb{1}_{\{|\mathbb{E}(\nu_n^t(u,\infty)) - k_n(t)\mathbb{E}\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})|>\varepsilon/3\}}. && (3.61)\end{aligned}$$

Observe that by a first order Chebychev inequality,

$$(3.59) \le |\mathbb{E}\nu_n^t(u,\infty) - k_n(t)\mathbb{E}\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})|. \tag{3.62}$$

By Lemmata 3.4, 3.5, and 3.7, (3.62) tends to zero as $n \to \infty$. For the same reason, (3.61) is equal to zero for large enough n. To bound (3.60), we calculate the variance of $k_n(t)\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})$. As in the proof of Proposition 3.8 we use Lemma 3.6, but take into account that there can only be contributions to the left hand side of (3.32) if $i,j \in \mathcal{I}_{\theta_n}$. This gives us the additional factor $(\gamma_n^2\rho_n)^2$ in (3.53). Therefore the variance of $k_n(t)\mathcal{E}_{\pi_n}G_n(u,H^0,\mathcal{I}_{\theta_n})$ is bounded above by $C(\gamma_n\rho_n)^2 n^{1-p/2}$ which, for all $p \ge 2$, vanishes as $n \to \infty$. Hence, we have proved Proposition 3.9. $\square$

3.3. **Second part of Condition (2-1).** We proceed as in Section 3.4 in [6] to verify the second part of Condition (2-1) . With the same notation as in (1.13), we define for $u > 0$ and $t > 0$

$$\begin{aligned} \widetilde{\eta}_n^t(u) \;&\equiv\; k_n(t)n^{-1}\sum_{x\in\Sigma_n}\left(Q_n^u(x)\right)^2 \;, && (3.63)\\ \eta_n^t(u) \;&\equiv\; k_n(t)\sum_{x\in\Sigma_n}\sum_{x'\in\Sigma_n}\mu_n(x,x')Q_n^u(x)Q_n^u(x')\;, && (3.64)\end{aligned}$$

where $\mu_n(\cdot,\cdot)$ is the uniform distribution on pairs $(x,x') \in \Sigma_n^2$ that are at distance 2 apart, i.e.

$$\mu_n(x,x') = \begin{cases} 2^{-n}\frac{2}{n(n-1)}, & \text{if } \mathrm{dist}(x,x') = 2,\\ 0, & \text{else.}\end{cases} \tag{3.65}$$

We prove that the expectations of both (3.63) and (3.64) tend to zero. First and second order Chebychev inequalities then yield that the second part of Condition (2-1) holds in $\mathbb{P}$-probability, respectively $\mathbb{P}$-a.s.

Lemma 3.10. *For all $u > 0$ and $t > 0$*

$$\lim_{n\to\infty}\mathbb{E}\widetilde{\eta}_n^t(u) = \lim_{n\to\infty}\mathbb{E}\eta_n^t(u) = 0\;. \tag{3.66}$$

Proof. We show that $\lim_{n\to\infty} \mathbb{E}\eta_n^t(u) = 0$. The assertion for $\widetilde{\eta}_n^t(u)$ is proved similarly. Let

$$\bar{Q}_n^u(x) \equiv \mathcal{P}_x\left(\sum_{j=1}^{\theta_n} \lambda_n^{-1}(J_n(j))e_{n,j} \le c_n u^{1/\alpha_n}\right). \tag{3.67}$$

Rewrite (3.64) in the following way

$$\begin{aligned} & k_n(t)\sum_{x\in\Sigma_n}\sum_{x'\in\Sigma_n}\mu_n(x,x')\left(1-\bar{Q}_n^u(x)\right)\left(1-\bar{Q}_n^u(x')\right) \\ = \;& k_n(t)\left[1-\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\left(\bar{Q}_n^u(x)+\bar{Q}_n^u(x')-\bar{Q}_n^u(x)\bar{Q}_n^u(x')\right)\right] \\ = \;& k_n(t)\left[1-2\sum_{x\in\Sigma_n}\pi_n(x)\bar{Q}_n^u(x)+\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\bar{Q}_n^u(x)\bar{Q}_n^u(x')\right]. \end{aligned} \tag{3.68}$$

To shorten notation, write

$$K_n^u \equiv \mathcal{P}_{\pi_n}\left(\max_{i\in\{\overline{\theta}_n,\dots,\theta_n\}} e^{\sqrt{n}\beta_n H^0(i)}e_{n,i} > c_n u^{1/\alpha_n}\,\middle|\,\mathcal{F}^J\right) = \sum_{x\in\Sigma_n}2^{-n}K_n^u(x), \tag{3.69}$$

where $\overline{\theta}_n \equiv 2n\log n$ and

$$K_n^u(x) \equiv \mathcal{P}_x\left(\max_{i\in\{\overline{\theta}_n,\dots,\theta_n\}} e^{\sqrt{n}\beta_n H^0(i)}e_{n,i} > c_n u^{1/\alpha_n}\,\middle|\,\mathcal{F}^J\right). \tag{3.70}$$

Using the bound $\bar{Q}_n^u(x) \le \mathcal{E}_x(1-K_n^u(x)) \equiv \mathcal{E}_x\bar{K}_n^u(x)$, $x\in\Sigma_n$, and taking expectation with respect to the random environment we obtain that

$$\begin{aligned} \mathbb{E}\eta_n^t(u) \;&\le\; k_n(t)-2\left(k_n(t)-\mathbb{E}\nu_n^t(u,\infty)\right) &(3.71)\\ &+\; k_n(t)\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\mathbb{E}\left[\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x')\right]. &(3.72)\end{aligned}$$

For $\bar{G}_n^u \equiv \mathcal{P}_{\pi_n}\left(\max_{i\in[\theta_n]} e^{\sqrt{n}\beta_n H^0(i)}e_{n,i} \le c_n u^{1/\alpha_n}\right)$ observe that

$$(3.71) \le k_n(t) - 2k_n(t)\mathbb{E}\bar{G}_n^u. \tag{3.73}$$

We add and subtract $\mathbb{E}\mathcal{E}_{\pi_n}(1-K_n^u) \equiv \mathbb{E}\mathcal{E}_{\pi_n}\bar{K}_n^u$ as well as

$$\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\mathbb{E}\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x'). \tag{3.74}$$

Re-arranging the terms and using the bound from (3.73) we see that $\mathbb{E}\eta_n^t(u)$ is bounded from above by

$$\begin{aligned} & 2k_n(t)\left(\mathbb{E}\bar{K}_n^u-\mathbb{E}\bar{G}_n^u\right) &(3.75)\\ +\;& k_n(t)\sum_{x,x'}\mu_n(x,x')\mathbb{E}\mathcal{E}_x K_n^u(x)\mathbb{E}\mathcal{E}_{x'}K_n^u(x') &(3.76)\\ +\;& k_n(t)\sum_{x,x'}\mu_n(x,x')\left(\mathbb{E}\left[\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x')\right]-\mathbb{E}\mathcal{E}_x\bar{K}_n^u(x)\mathbb{E}\mathcal{E}_{x'}\bar{K}_n^u(x')\right). &(3.77)\end{aligned}$$

From Proposition 3.3 we conclude that (3.75) and (3.76) are of order $O\left(\frac{\log n}{n}\right)$ and $O\left(\theta_n a_n^{-1}\right)$ respectively. To control (3.77) we use the normal comparison theorem (Lemma 3.6) for the processes V^0 and V^1 as in Proposition 3.8. However, due to the fact that we are looking at the chain after $\bar{\theta}_n$ steps, the comparison is simplified. More precisely, let $\mathcal{A}_n \equiv \left\{\forall\bar{\theta}_n \le i \le \theta_n : \operatorname{dist}(J_n(i),J_n'(i)) > n(1-\rho(n))\right\} \subset \mathcal{F}^J\times\mathcal{F}^{J'}$, where $\rho(n)$ is of the order of $\sqrt{n^{-1}\log n}$. Then, on $\mathcal{A}_n$, by Lemma 3.6 and the estimates from (3.35),

$$\mathbb{E}\left[\bar{K}_n^u(x)\bar{K}_n^u(x')\right]-\mathbb{E}\bar{K}_n^u(x)\mathbb{E}\bar{K}_n^u(x') \le 2\gamma_n^{-2}\sum_{\substack{1\le i\le\theta_n\\ \theta_n+1\le j\le 2\theta_n}}\Delta_{ij}^0 e^{-\gamma_n^2 n(1+\Delta_{ij}^0)^{-1}} \le O(\theta_n^2 a_n^{-2}). \tag{3.78}$$

Moreover, on $\mathcal{A}_n^c$,

$$\mathbb{E}\left[\bar{K}_n^u(x)\bar{K}_n^u(x')\right] - \mathbb{E}\bar{K}_n^u(x)\mathbb{E}\bar{K}_n^u(x') \leq O(a_n^{-1}). \tag{3.79}$$

But in Lemma 3.7 from [6] it is shown that for a specific choice of $\rho(n)$ and every $x \in \Sigma_n$

$$\begin{aligned} P\left(\mathcal{A}_n \middle| \operatorname{dist}(J_n(0), J_n'(0)) = 2\right) &\geq 1 - n^{-8} \\ P_x\left(\mathcal{A}_n^c\right) &\leq n^{-4}. \end{aligned} \tag{3.80}$$

Therefore we obtain that $\lim_{n\to\infty} \mathbb{E}\eta_n^t(u) = 0$. □

Remark. Lemma 3.10 immediately implies that the second part of Condition (2-1) holds in $\mathbb{P}$-probability. To show that it is satisfied $\mathbb{P}$-almost surely for $p > 5$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c < \frac{1}{4}$ it suffices to control the variance of (3.75). We use the same concentration results as in Proposition 3.8 to obtain that the variance of $k_n(t)(\bar{K}_n^u - \bar{G}_n^u)$, which is given by

$$k_n^2(t)\left[\mathbb{E}\left(\bar{K}_n^u - \mathbb{E}\bar{K}_n^u\right)^2 + \mathbb{E}\left(\bar{G}_n^u - \mathbb{E}\bar{G}_n^u\right)^2 - 2\left(\mathbb{E}\bar{G}_n^u\bar{K}_n^u - \mathbb{E}\bar{G}_n^u\mathbb{E}\bar{K}_n^u\right)\right], \tag{3.81}$$

is bounded from above by $C\gamma_n^{-2}n^{1-p/2}$.

3.4. **Condition (3-1).** We show that Condition (3-1) is $\mathbb{P}$- a.s. satisfied for all $\delta > 0$.

Lemma 3.11. *We have $\mathbb{P}$-a.s. that*

$$\limsup_{n\to\infty}\left(a_n\left(c_n\delta^{1/\alpha_n}\right)^{-1}\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(1))e_{n,1}\mathbb{1}_{\lambda_n^{-1}(J_n(1))e_{n,1}\leq c_n\delta^{1/\alpha_n}}\right)^{\alpha_n} < \infty, \quad \forall \delta > 0. \tag{3.82}$$

Proof. We begin by proving that for all $\delta > 0$, for n large enough,

$$\begin{aligned} \tfrac{a_n}{c_n\delta^{1/\alpha_n}}\mathcal{E}_{\pi_n}\mathbb{E}\lambda_n^{-1}(J_n(1))e_{n,1}\mathbb{1}_{\lambda_n^{-1}(J_n(1))e_{n,1}\leq c_n\delta^{1/\alpha_n}} &= \textstyle\sum_{x\in\Sigma_n} 2^{-n}\mathbb{E}Y_{n,\delta}(x) \\ &\leq 4(\delta\gamma_n\beta_n)^{-1}, \end{aligned} \tag{3.83}$$

where $Y_{n,\delta}(x) \equiv a_n\left(c_n\delta^{1/\alpha_n}\right)^{-1}\lambda_n^{-1}(x)e_{n,1}\mathbb{1}_{\lambda_n^{-1}(x)e_{n,1}\leq c_n\delta^{1/\alpha_n}}$, for $x \in \Sigma_n$.

For $x \in \Sigma_n$ we have that

$$\begin{aligned} \mathbb{E}Y_{n,\delta}(x) &= a_n(c_n\delta^{1/\alpha_n})^{-1}(2\pi)^{-1/2}\int_0^\infty dy\int_{-\infty}^{y_n} dz\; ye^{-y-\frac{z^2}{2}+\beta_n\sqrt{n}z} \\ &= a_n(c_n\delta^{1/\alpha_n})^{-1}(2\pi)^{-1/2}\int_0^\infty dy\int_{\beta_n\sqrt{n}-y_n}^{\infty} dz\; ye^{-y+\frac{\beta_n^2 n}{2}-\frac{z^2}{2}}, \end{aligned} \tag{3.84}$$

where $y_n \equiv (\sqrt{n}\beta_n)^{-1}\left(\log c_n + \frac{\beta_n}{\gamma_n}\log\delta - \log y\right)$ for $y > 0$. In order to use estimates on Gaussian integrals, we divide the integration area over y into $y \leq n^2$ and $y > n^2$.

For $y > n^2$, there exists a constant $C' > 0$ such that

$$(2\pi)^{-1/2}a_n(c_n\delta^{1/\alpha_n})^{-1}\int_{n^2}^\infty dy\int_{-\infty}^{y_n} dz\; ye^{-y-\frac{z^2}{2}+\beta_n\sqrt{n}z} \leq C'a_n n^4 e^{-n^2}, \tag{3.85}$$

which vanishes as $n \to \infty$.

Let $y \leq n^2$. By definition of c_n we have $\beta_n\sqrt{n} - y_n = \sqrt{n}\beta_n\left(1 - \frac{\gamma_n}{\beta_n} - \frac{\log\delta}{\gamma_n\beta_n n} + \frac{\log y}{\beta_n^2 n}\right)$. Since $\alpha_n \downarrow 0$ as $n \to \infty$, it follows that for n large enough $\beta_n\sqrt{n} - y_n > 0$. But then, since $\mathbb{P}(Z > z) \leq (\sqrt{2\pi})^{-1}z^{-1}e^{-z^2/2}$ for any $z > 0$ and Z being a standard Gaussian,

$$\int_0^{n^2} dy\int_{-y_n+\beta_n\sqrt{n}}^\infty dz\; ye^{-y+\frac{\beta_n^2 n}{2}-\frac{z^2}{2}} \leq \int_0^{n^2} dy\frac{ye^{-y}}{\beta_n\sqrt{n}-y_n}e^{\frac{\beta_n^2 n}{2}-\frac{(\beta_n\sqrt{n}-y_n)^2}{2}}. \tag{3.86}$$

Plugging in the definition of a_n and c_n, (3.85) and (3.86) yield that, for n large enough, up to a multiplicative error that tends to 1 as $n \to \infty$ exponentially fast,

$$\begin{aligned}
(3.84) \quad &\leq \int_0^{n^2} dy\, y^{\alpha_n} e^{-y} (\gamma_n \beta_n \delta)^{-1} \left(1 - \tfrac{\gamma_n}{\beta_n} - \tfrac{\log \delta}{n \gamma_n \beta_n} + \tfrac{\log y}{\beta_n^2 n}\right)^{-1} e^{2 \log \delta \log n (n \gamma_n \beta_n)^{-1}} \\
&\leq 2 \int_0^{n^2} dy\, y^{\alpha_n} e^{-y} (\gamma_n \beta_n \delta)^{-1} \\
&\leq 2 \Gamma\left(1 + \tfrac{\gamma_n}{\beta_n}\right) (\gamma_n \beta_n \delta)^{-1}, \qquad (3.87)
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function. Since $\Gamma(1 + \alpha_n) \leq 1$ for $\alpha_n \leq 1$, the claim of (3.83) holds true for all $\delta > 0$ for n large enough.

Lemma 3.10 from [6] yields that for all $\delta > 0$ there exists $\kappa > 0$ such that

$$\mathbb{E}\left(\mathcal{E}_{\pi_n} Y_{n,\delta}\right)^2 - \left(\mathbb{E}\mathcal{E}_{\pi_n} Y_{n,\delta}\right)^2 \leq a_n^2 \left(c_n \delta^{1/\alpha_n}\right)^{-2} n^{1-p/2} \leq e^{-n^\kappa}, \qquad (3.88)$$

where $\mathcal{E}_{\pi_n} Y_{n,\delta} \equiv \sum_{x \in \Sigma_n} 2^{-n} Y_{n,\delta}(x)$. For all $\delta > 0$ there exists by Borel-Cantelli Lemma a set $\Omega(\delta)$ with $\mathbb{P}(\Omega(\delta)) = 1$ such that on $\Omega(\delta)$, for all $\varepsilon > 0$ there exists $n' \in \mathbb{N}$ such that

$$\mathcal{E}_{\pi_n} Y_{n,\delta} \leq 4 \left(\gamma_n \beta_n \delta\right)^{-1} + \varepsilon, \quad \forall n \geq n'. \qquad (3.89)$$

Setting $\Omega^\tau \equiv \bigcap_{\delta \in \mathbb{Q} \cap (0,\infty))} \Omega(\delta)$, we have $\mathbb{P}(\Omega^\tau) = 1$.

Let $\delta > 0$ and $\varepsilon > 0$. We can always find $\delta' \in \mathbb{Q}$ such that $\delta \leq \delta' \leq 2\delta$. Note that $Y_{n,\delta}$ is increasing in δ. Moreover, by (3.89) there exists $n' = n'(\delta', \varepsilon)$ such that on Ω^τ and for $n \geq n'$

$$\left(\mathcal{E}_{\pi_n} Y_{n,\delta}\right)^{\alpha_n} \leq \left(\mathcal{E}_{\pi_n} Y_{n,\delta'}\right)^{\alpha_n} \leq \left(4 \left(\gamma_n \beta_n \delta'\right)^{-1} + \varepsilon\right)^{\alpha_n} \leq 4 \left(\gamma_n \beta_n \delta'\right)^{-\alpha_n}. \qquad (3.90)$$

Since $(\gamma_n \beta_n)^{-\alpha_n} \downarrow 1$ as $n \to \infty$, we obtain the assertion of Lemma 3.11. $\square$

3.5. **Proof of Theorem 1.4.** We are now ready to conclude the proof of Theorem 1.4.

First let $p > 5$ and $\gamma_n = n^{-c}$ for $c \in \left(0, \frac{1}{2}\right)$, or $p = 5$ and $c > \frac{1}{4}$. Then we know by Propositions 3.3 and 3.8 that for all $u > 0$ there exists a set $\Omega(u)$ with $\mathbb{P}(\Omega(u)) = 1$, such that on $\Omega(u)$

$$\lim_{n \to \infty} \nu_n^t(u, \infty) = K_p t u^{-1}, \quad \forall t > 0. \qquad (3.91)$$

The mapping that maps u to $\nu_n^t(u,\infty)$ is decreasing on $(0,\infty)$ and its limit, u^{-1}, is continuous on the same interval. Therefore, setting $\Omega_1^\tau = \bigcap_{u \in (0,\infty) \cap \mathbb{Q}} \Omega(u)$, we have $\mathbb{P}(\Omega_1^\tau) = 1$ and (3.91) holds true for all $u > 0$ on Ω_1^τ. By the same arguments and the results in Section 3.3 there also exists a subset Ω_2^τ with full measure and such that the second part of Condition (2-1) holds on Ω_2^τ.

Condition (3-1) holds $\mathbb{P}$-a.s. by Lemma 3.11. Finally, we are left with the verification of Condition (0) for the invariant measure $\pi_n(x) = 2^{-n}$, $x \in \Sigma_n$. For $v > 0$, we have that

$$\sum_{x \in \Sigma_n} 2^{-n} e^{-v^{\alpha_n} c_n \lambda_n(x)} = \sum_{x \in \Sigma_n} 2^{-n} \mathcal{P}_{\pi_n}\left(\lambda_n^{-1}(x) e_{n,1} > c_n v^{\alpha_n}\right). \qquad (3.92)$$

By similar calculations as in (3.87), we see that, for n large enough and $x \in \Sigma_n$,

$$\mathbb{E}\mathcal{P}_{\pi_n}\left(\lambda_n^{-1}(x) e_{n,1} > c_n v^{\alpha_n}\right) \sim a_n^{-1} \gamma_n^2 v^{-1}, \qquad (3.93)$$

which tends to zero as $n \to \infty$. By a first order Chebychev inequality we conclude that for all $v > 0$ Condition (0) is satisfied $\mathbb{P}$-a.s. As before, by monotonicity and continuity, this implies that Condition (0) holds $\mathbb{P}$-a.s. for all $v > 0$. This proves Theorem 1.4 in this case.

For $p = 2, 3, 4$ and $c \in \left(0, \frac{1}{2}\right)$ or $p = 5$ and $c \geq \frac{1}{4}$, we know from Propositions 3.3, 3.8, and Section 3.3 that Condition (2-1) is satisfied in $\mathbb{P}$-probability, whereas Condition (0) and (3-1) hold $\mathbb{P}$-a.s. This concludes the proof of Theorem 1.4.

3.6. **Proof of Theorem 1.5.** We use Theorem 1.4 to prove the claim of Theorem 1.5. By the same arguments as in the proof of Theorem 1.5 in [6], we obtain that for $t > 0$, $s > 0$, and $\varepsilon \in (0, 1)$ the correlation function $\mathcal{C}_n^\varepsilon(t, s)$ can, with very high probability and $\mathbb{P}$- a.s., be approximated by

$$\begin{aligned} \mathcal{C}_n^\varepsilon(t, s) &= (1 - o(1))\, \mathcal{P}_{\pi_n}(\mathcal{R}_n \cap (t^{\alpha_n}, (t + s)^{\alpha_n}) = \emptyset) \\ &= (1 - o(1))\, \mathcal{P}_{\pi_n}(\mathcal{R}_{\alpha_n} \cap (t, t + s) = \emptyset), \end{aligned} \tag{3.94}$$

where $\mathcal{R}_n$ is the range of the blocked clock process S_n^b and $\mathcal{R}_{\alpha_n}$ is the range of $\left(S_n^b\right)^{\alpha_n}$. By Theorem 1.4 we know that $\left(S_n^b\right)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$, $\mathbb{P}$-a.s. for $p > 5$ if $c \in (0, \frac{1}{2})$, $p = 5$ if $c < \frac{1}{4}$, and in $\mathbb{P}$-probability else. By Proposition 4.8 in [15] we know that the range of M_ν is the range of a Poisson point process ξ' with intensity measure $\nu'(u, \infty) = \log u - \log K_p$. Thus, writing $\mathcal{R}_M$ for the range of M_ν, we get that

$$\mathcal{P}(\mathcal{R}_M \cap (t, t + s) = \emptyset) = \mathcal{P}(\xi'(t, t + s) = 0) = e^{-\nu'(t,t+s)} = \tfrac{t}{t+s}. \tag{3.95}$$

The claim of Theorem 1.5 follows.

4. Appendix

In the appendix we state and prove a lemma that is needed in the proof of Lemma 3.7.

Lemma 4.1. *Let $D_{ij} = \mathrm{dist}(J_n(i), J_n(j))$ and $\Delta_d^0 = (1 - 2dn^{-1})^p$. For any $\eta > 0$ there exists a constant $\bar{C} < \infty$ such that, for n large enough and $d \in \{0, \dots, n\}$,*

$$k_n(t) \textstyle\sum_{\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} |\Delta_d^0 - \Delta_{ij}^1| \leq \bar{C} t a_n \frac{d^2}{v_n n} \mathbb{1}_{d \leq v_n}, \tag{4.1}$$

$$k_n(t) \textstyle\sum_{\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq \bar{C} t \frac{a_n \exp\left(\eta \gamma_n^2 \min\{d, n-d\}\right)}{v_n \gamma_n^2}. \tag{4.2}$$

Proof. We use ideas from Section 3 in [1] and Section 4 in [2] and write the distance process $D_{ij} = \mathrm{dist}(J_n(i), J_n(j))$ as the Ehrenfest chain $Q_n = \{Q_n(k) : \ k \in \mathbb{N}\}$, which is a birth-death process with state space $\{0, \dots, n\}$ and transition probabilities $p_{k,k-1} = 1 - p_{k,k+1} = \frac{k}{n}$ for $k \in \{0, \dots, n\}$. Denote by P_k the law and E_k the expectation of Q_n starting in k. Let moreover $T_d = \inf\{k \in \mathbb{N} : \ Q_n(k) = d\}$. By the Markov property of J_n, we have under P_0, in distribution, that

$$\mathrm{dist}(J_n(0), J_n(k)) \overset{d}{=} \mathrm{dist}(J_n(j), J_n(j + k)) \overset{d}{=} Q_n(k) \ , \quad \forall j, k \geq 0. \tag{4.3}$$

Recall for the proof of (4.1) that if $\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor$, we have that $\Delta_{ij}^1 \leq \Delta_{i,j}^0$. Moreover, since for such i, j necessarily $|i - j| \leq v_n$ we have that $D_{ij} \leq v_n$. Thus, let $d \in \{1, \dots, v_n\}$. By Lemma 4.2 in [1] we deduce that there exists a constant $C < \infty$, independent of d, such that

$$k_n(t) \textstyle\sum_{\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq C t a_n \ . \tag{4.4}$$

Moreover,

$$\left(\Delta_d^0 - \Delta_{ij}^1\right) = \left(1 - \tfrac{2d}{n}\right)^p - \left(1 - \tfrac{2p|i-j|}{n}\right) = \tfrac{2p}{n} \left(|i - j| - d\right) + O\left(\tfrac{d^2}{n^2}\right). \tag{4.5}$$

Therefore the main contributions in (4.1) are of the form

$$\begin{aligned}\textstyle\sum_{\lfloor i/v_n\rfloor=\lfloor j/v_n\rfloor}^{\theta_n}\left(|i-j|-d\right)E_{\pi_n}\mathbb{1}_{D_{ij}=d} &= v_n\textstyle\sum_{i=1}^{\lfloor\theta_n/v_n\rfloor}\sum_{j=i+1}^{i+v_n}\left(j-i-d\right)E_{\pi_n}\mathbb{1}_{D_{ij}=d}\\ &= v_n\textstyle\sum_{i=1}^{\lfloor\theta_n/v_n\rfloor}\sum_{j=1}^{v_n}E_0\mathbb{1}_{Q_n(j)=d}\left(j-d\right).\end{aligned}\tag{4.6}$$

Setting $Z\equiv\sum_{j=1}^{v_n}\mathbb{1}_{Q_n(j)=d}\left(j-d\right)$, (4.6) is nothing but θ_nE_0Z. It is shown in [2] (page 107-108) that there exists a constant $C<\infty$, independent of d, such that

$$\begin{aligned}E_0Z &\leq CE_0\left(T_d-d\right)\mathbb{1}_{T_d<v_n}\\ &\leq C\left(E_0T_d-dP_0\left(T_d<v_n\right)\right)\leq C\left(E_0T_d-d\left(1-v_n^{-1}E_0T_d\right)\right)\ ,\end{aligned}\tag{4.7}$$

where the last inequality is obtained by a first order Chebychev inequality. To calculate E_0T_d we use the following classical formulas (see e.g. [12], Chapter 2.5)

$$E_0T_d = \textstyle\sum_{l=1}^{d}E_{l-1}T_l,\qquad\text{where}\tag{4.8}$$

$$E_{l-1}T_l = \tfrac{1}{p_{l,l-1}}\textstyle\prod_{i=1}^{l}\tfrac{p_{i,i-1}}{p_{i-1,i}}\left(1+\sum_{j=1}^{l-1}\prod_{k=1}^{j}\tfrac{p_{k,k-1}}{p_{k-1,k}}\right).\tag{4.9}$$

Plugging in the transition probabilities, we obtain for all $l\leq d$,

$$\begin{aligned}E_{l-1}T_l &= \tfrac{n}{l}\left(\textstyle\prod_{i=1}^{l}\tfrac{i}{n-i+1}+\sum_{j=1}^{l-1}\prod_{k=j+1}^{l}\tfrac{k}{n-k+1}\right)\\ &= \tfrac{n}{l}\textstyle\sum_{j=0}^{l-1}\prod_{k=j+1}^{l}\tfrac{k}{n-k+1}.\end{aligned}\tag{4.10}$$

For any $l\leq d$ and $0\leq j\leq l-1$ we have that

$$\tfrac{n}{l}\textstyle\prod_{k=j+1}^{l}\tfrac{k}{n-k+1}\leq\tfrac{n}{d}\prod_{k=j+1}^{l}\tfrac{d}{n-d}\ .\tag{4.11}$$

In view of (4.8) we get that

$$E_0T_d\leq\textstyle\sum_{l=1}^{d}\tfrac{1}{1-2dn^{-1}}\left(1-\left(\tfrac{d}{n-d}\right)^l\right)\leq\tfrac{d}{(1-2dn^{-1})}.\tag{4.12}$$

But then, since $\frac{d}{n}\downarrow 0$ as $n\rightarrow\infty$ and $d\leq v_n$, there exists a constant $C'<\infty$, independent of d, such that

$$E_0Z\leq C'\tfrac{d^2}{v_n}.\tag{4.13}$$

Together with (4.4) and (4.5) this concludes the proof of (4.1).

For the proof of (4.2) we distinguish several cases. If $\|d\|\equiv\min\{d,n-d\}>(\log n)^{1+\varepsilon}\gamma_n^{-2}$ for some fixed $\varepsilon>0$ then the claim of (4.2) is deduced from the bound

$$k_n(t)\textstyle\sum_{\lfloor i/v_n\rfloor\neq\lfloor j/v_n\rfloor}^{\theta_n}E_{\pi_n}\mathbb{1}_{D_{ij}=d}\leq a_nt\theta_n\ll a_nt\tfrac{e^{\eta\|d\|\gamma_n^2}}{v_n\gamma_n^2}.\tag{4.14}$$

Assume next that $\|d\|\leq(\log n)^{1+\varepsilon}\gamma_n^{-2}$. It is shown in [2], (page 111-112), that in this case one can neglect values of d such that $d\geq\frac{n}{2}$. Thus, let $d\leq(\log n)^{1+\varepsilon}\gamma_n^{-2}$. Note that

$$k_n(t)\textstyle\sum_{\lfloor i/v_n\rfloor\neq\lfloor j/v_n\rfloor}^{\theta_n}E_{\pi_n}\mathbb{1}_{D_{ij}=d}\leq k_n(t)\sum_{k=0}^{\theta_n}\sum_{m=j_k}^{\theta_n}E_{\pi_n}\mathbb{1}_{D_{k,k+m}=d}\ ,\tag{4.15}$$

where $j_k=\inf\{i\in\mathbb{N}:\ \lfloor k/v_n\rfloor\neq\lfloor(k+i)/v_n\rfloor\}$.

We further distinguish the cases $j_k\leq 2d$ and $j_k>2d$. If $j_k\leq 2d$ then, setting $Z_{j_k}(d)\equiv\sum_{m=j_k}^{\theta_n}\mathbb{1}_{D_{k,k+m}=d}$, we have $Z_{j_k}(d)\leq Z_0(d)$. It is shown on page 685 in [1] that there exists $C<\infty$, independent of d, such that $E_0Z_0(d)\leq C$. Since moreover $|\{k\in\{1,\ldots,\theta_n\}:j_k\leq 2d\}|\leq 2\frac{d\theta_n}{v_n}$, we know that for all $\eta>0$ there exists $C'<\infty$ such that

$$k_n(t)\textstyle\sum_{k=0}^{\theta_n}\sum_{m=j_k}^{\theta_n}E_{\pi_n}\mathbb{1}_{D_{ij}=d}\leq Ct\tfrac{a_nd}{v_n}\leq C't\tfrac{a_ne^{\eta\gamma_n^2\|d\|}}{v_n\gamma_n^2}.\tag{4.16}$$

Let $j_k > 2d$, i.e. in particular $Z_{j_k}(d) \leq Z_{2d}(d)$. By the Markov property and by Lemma 4.2 in [1] we obtain that there exists $C < \infty$ such that

$$E_0 Z_{2d}(d) \leq P_0(T_d \in (2d, \theta_n)) \left(1 + E_d \left(\sum_{k=1}^{\theta_n} \mathbb{1}_{Q_n(k)=d}\right)\right) \leq C P_0(T_d \in (2d, \theta_n)). \tag{4.17}$$

The probability that Q gets from 0 to d after $2d$ steps is bounded by the probability that it takes at least d steps to the left, i.e.

$$P_0(T_d \in (2d, \theta_n)) \leq \binom{2d}{d} \left(\frac{d}{n}\right)^d \leq 2d \left(\frac{4d}{n}\right)^d \ll \frac{d}{v_n}. \tag{4.18}$$

The claim follows as in (4.16). This finishes the proof of (4.2). $\square$

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